Asset prices and wealth inequality in a simple model with idiosyncratic shocks*

Precios de activos y desigualdad de la riqueza en un modelo simple con shocks idiosincráticos

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Abstract

This paper analytically solves a heterogeneous agent model with idiosyncratic shocks to marginal utility of consumption and explores the effects of the borrowing constraint on the price of the asset, the composition of borrowers and lenders in the credit market, and wealth inequality. Results are derived in a stylized model and in a pedagogical fashion.

Key words: *Asset prices, borrowing constraints, wealth inequality, heterogeneous agents.*

JEL Classification: *E00, E44, G10.*

Resumen

Este documento presenta la solución analítica de un modelo de agentes heterogéneos quienes reciben shocks idiosincráticos en la utilidad marginal del consumo, y explora los efectos de la restricción de crédito sobre el precio de los activos, la composición entre demandantes y oferentes en el mercado de crédito, y sobre la desigualdad de la riqueza. Los resultados son derivados en un modelo estilizado y con un enfoque pedagógico.

Palabras clave: *Precios de activos, restricciones de crédito, desigualdad de la riqueza, agentes heterogéneos.*

Clasificación JEL: *E00, E44, G10.*

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1. Introduction

Shocks to marginal utility of consumption, or "urgencies to consume," are one way to motivate differential demand of an asset for liquidity purposes in a model economy.[1] Characterizing existence under borrowing constraints and other parameters and their effects on different objects of interest often requires the use of global numerical methods even in simple economies of this sort. The contribution of this paper is to show, in a pedagogical and analytical fashion, the existence and uniqueness of equilibrium and how the borrowing constraint affects the price of the asset, the distribution of individuals by asset, the number of borrowers and lenders, and wealth inequality.

The effects of tightening the borrowing constraint are an increase in the equilibrium price of the asset, fewer lenders and more borrowers, and a decrease in wealth inequality.[2] It is possible to find analytical results due to two assumptions. The first assumption is that utility is linear in consumption. This assumption is used by Taub (1994) in examining environments with money and credit. While the author is able to derive existence results, his environment allows for only partial characterization of the effect of the borrowing constraint on the price of the asset. This is because he works with the "natural borrowing constraint" in the sense of Aiyagari (1994), in which borrowing is limited by the present-value budget balance. In contrast, I use an ad-hoc borrowing constraint that allows for flexibility to find the effect of changing borrowing limits not only on the asset price but on different objects of interest. The second assumption not used in Taub (1994) is to consider the Pareto distribution for the preferences shock. This distribution has good empirical properties, as documented in Wen (2015). But I use it mainly because of its analytical convenience, as it allows for derivation of closed-form solutions for all equilibrium objects of the model. Then, studying the effects of changing the borrowing constraint is straightforward.

Other Related Literature

The model presented in this paper fits into the class of heterogeneous agent models where idiosyncratic shocks induce credit arrangements in equilibrium. One class of models of this sort belongs to the Bewley (1977) tradition, such as Hugget (1993), Aiyagari (1994), and Krusell and Smith (1998). These papers consider shocks to income, not to marginal utility of consumption. Achdou et al. (2014) study Bewley-type models in continuous time, making some progress on analytical results instead of relying on numerical methods. Krusell et al. (2011) also obtain analytical results for asset prices in an environment similar to that of Hugget (1993) in which they consider "maximally tight" borrowing constraints. These papers do not show both existence and uniqueness of equilibrium, nor do they explore the effects of the borrowing constraint on the number of lenders and borrowers and wealth inequality.

[1] See, for example: Lucas (1980), Taub (1988), Farhi Werning (2007) and Wen (2015).

[2] It will also be shown that the price of the asset also increases when the mean and variance of the underlying distribution of shocks increase.

Lucas (1980), Taub (1988), Taub (1994), Wen (2015), Lucas (1992), and Atkenson and Lucas (1992) consider shocks to marginal utility of consumption. They explore different questions, ranging from monetary policy to efficiency under private information. None of these contributions uses a simple, although very stylized, pedagogical model such as the one developed here to explore the effects of borrowing constraints on the objects of interest described above.

The paper is organized as follows: Section 2 presents the model, Section 3 presents the solution and analysis, and Section 4 concludes. All proofs are presented in the appendix.

2. The Model

The economy is populated by a measure one of infinitely lived individuals who receive a constant perishable endowment y in each period and trade claims to future consumption at price q, such that current consumption is $c=a+y-qa'\geq 0$. They are exogenously constrained to not borrow more than $b>0$ units of the asset, $a'\geq -b$. Each period they face the urgency to consume $\theta \in \Theta \equiv [\underline{\theta}, \overline{\theta}]$, a draw from CDF $F(\theta)$ defined over Θ that affects their marginal utility of consumption. The value function for an agent with state (a, θ) is defined as:[3]

$$v(a,\theta) = \max_{-b\leq a' \leq \frac{a+y}{q}} \left[\theta(a+y-qa') + \beta \int_{\Theta} v(a',\theta')dF(\theta') \right] \tag{2.1}$$

where a' is constrained to be greater than the borrowing limit $-b$ and not so large as to make consumption negative.

The distribution of individuals by asset is denoted by $\Psi(a)$. If a solution to (2.1) exists, it will deliver a policy function for next-period asset $g(a,\theta)$ and hence for consumption $c(a,\theta) = a + y - qg(a,\theta)$. The requirement for an invariant distribution of individuals by asset is:

$$\Psi(a') = \int_{\Theta} \int_{\mathcal{A}(a')} d\Psi(a)\, dF(\theta) \tag{2.2}$$

where: $\mathcal{A}(a') = [a:\ a \geq -b,\ \theta \in \Theta,\ g(a,\theta) \leq a']$.

Finally, given that the asset is in zero net supply, any equilibria must satisfy:

$$\int_{\Theta} \int_{\mathcal{A}(a')} g(a,\theta)\, d\Psi(a)\, dF(\theta) = 0 \tag{2.3}$$

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[3] The value function and other objects in the model are functions of q. To simplify the notation, I omit explicit dependence of such objects on q. Later I will find the equilibrium value of q, denoted by q^*. Since the environment is stationary, this price is constant and individuals take it as given when optimizing.

Definition of Equilibrium: *A stationary equilibrium is an asset price q^* and a distribution Ψ such that:*

1. *Individuals solve the problem (2.1)*
2. *Markets clear, equation (2.3) is satisfied*
3. *The distribution is invariant, equation (2.2) is satisfied.*

3. Analysis

I start by showing the existence of the value function defined in (2.1) and its associated policy functions.

Proposition 1. There exists a threshold $z \in int(\Theta)$ implicitly defined by:

$$z = \frac{\beta}{q - \beta F(z)} \int_z^{\bar{\theta}} \theta \, dF(\theta) \tag{3.1}$$

such that policy functions are given by:

$$g(a,\theta) = \begin{cases} \dfrac{a+y}{q} & \text{if } \theta \leq z \\ -b & \text{if } \theta > z \end{cases} \quad \text{and} \quad c(a,\theta) = \begin{cases} 0 & \text{if } \theta \leq z \\ a+y+qb & \text{if } \theta > z \end{cases} \tag{3.2}$$

and the value function exists and its closed-form solution is:

$$v(a,\theta) = \begin{cases} \left[q^2 b + (y - b\beta)q + (y+a)(1-\beta)\right] \dfrac{z}{1-\beta} & \text{if } \theta \leq z \\ \theta(y+a+qb) + \left[q^2 b + (y-b)q\right] \dfrac{z}{1-\beta} & \text{if } \theta > z \end{cases}. \tag{3.3}$$

The characterization in Proposition 1 shows that an individual who faces a small urgency to consume would save all of her resources, whereas if the urgency to consume is large, she would consume all of her income plus the maximum borrowing amount. The threshold value of θ, denoted by z, distinguishes between a large and small urgency.[4] If $q^* = \beta$, the result obtained would be identical to that of a representative-agent model–the asset value is the same as the discount factor–which cannot be ruled out yet since $q^* > \beta F(z)$ may be satisfied.[5] I now turn to the determination of the distribution $\Psi(a)$.

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[4] Note that in order for z to be well-defined, $q > \beta F(z)$ must hold. Up to this point, without having the model's equilibrium, it is not possible to verify this.

[5] We can rule out that in equilibrium $q^* < \beta$, because (3.1) would imply $z<0$, which is not possible.

Proposition 2. $\Psi(a)$, satisfying (2.2) exists:

$$\Psi\left(a_i\right) = 1 - F(z)^i, \quad i = 1,2,3,\ldots \tag{3.4}$$

with density:

$$d\Psi(a_i) = 1 - F(z)^i - \left[1 - F(z)^{i-1}\right] = \left[1 - F(z)\right]F(z)^{i-1}, \quad i = 1,2,3,\ldots \tag{3.5}$$

and the support $\{a_i\}_{i=1}^{\infty}$ is defined by:

$$a_i = -b\left(\frac{1}{q}\right)^{i-1} + \frac{y}{q-1}\left[1 - \left(\frac{1}{q}\right)^{i-1}\right], \quad i = 1,2,3,\ldots \tag{3.6}$$

The linear utility plus the i.i.d. assumption of the urgencies to consume lead to both a closed-form solution for the distribution as well as its special features. Since F depends on the value q, and the equilibrium q^* depends on b, the borrowing constraint has a direct effect on the shape of the distribution, both in the support and the number of agents at each point of the support. Depending on whether q^* is higher or lower than one, the support may be unbounded or bounded. I will examine this issue once the equilibrium asset price is found.

3.1. Equilibrium

With closed-form solutions in hand, an expression for market clearing in (2.3) can be written as:

$$b\left[1 - F(z)\right]q^2 - \left\{F(z)y + \left[1 - F(z)\right]b\right\}q + yF(z) = 0, \tag{3.7}$$

using both (3.5) and (3.6). To get this result, the following inequality was assumed to be satisfied:

$$F(z) < q. \tag{3.8}$$

Here I am employing a guess-and-verify strategy as conditions on the parameters such that (3.8) is satisfied will be found later.

To show the existence of equilibrium, one must show that there exists q^* and z such that both (3.7) and (3.1) are satisfied. I simplify this task by assuming that $F(\theta)$ is Pareto. Wen (2015) uses the same assumption in a related model and finds that it has some good empirical properties, but given the stylized nature of the present paper, I use the assumption for analytical convenience.

Assumption 1. θ is assumed to be distributed Pareto:[6]

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6 The Pareto distribution requires σ to be large enough for the mean and variance to exist. In particular. the mean exists if $\sigma > 1$, which is imposed from the outset. The variance exists if $\sigma > 2$; this is not imposed as it is not required for the results derived.

(3.9) $$F(\theta)=1-\theta^{-\sigma}, \quad \sigma>1, \quad \theta\in\Theta\equiv[1,\infty)$$

With Assumption 1, there is an explicit solution for z in (3.1):

(3.10) $$z=\left[\frac{\beta}{(q-\beta)(\sigma-1)}\right]^{1/\sigma}$$

Thus, there is a negative relationship between the threshold value z and the price of the asset q. Later, I will explain the intuition once the solution of the model is cast into a simple supply and demand framework.

To simplify the computation used to find the equilibrium, I assume that the borrowing constraint is a proportion of the period endowment: $b=ky$, where $k>0$. The next proposition shows existence and characterizes q^*.

Proposition 3. The equilibrium is unique and given by:

(3.11a) $$q^*=\beta\left[\frac{\beta k-1+\sqrt{(\beta k-1)^2+4\beta k\frac{\sigma}{\sigma-1}}}{2\beta k}\right],$$

for all parameter values such that:

(3.11b) $$0<k<\bar{k}\equiv\frac{\sigma-1+\beta}{(\sigma-1)(1-\beta)}.$$

Condition (3.11b) ensures that (3.8) is satisfied and puts a lower bound on the borrowing constraint $a'\geq -b=-ky$. The higher the value of σ, the higher the value of $\bar{k}$, which can be shown by direct differentiation.

The following claims further characterize the equilibrium q^*:

Claim 1. The following results hold:

1. $q^*>\beta$.
2. $\frac{\partial q^*}{\partial k}<0, \ \frac{\partial q^*}{\partial \sigma}<0.$
3. As $k\to 0,\ q^*\to\frac{\beta\sigma}{\sigma-1}$.
4. The minimum feasible value of q^* is attained when $k\to\bar{k}$.
5. $\frac{\partial^2 q^*}{\partial k\,\partial\sigma}>0.$

The first claim states that the asset has a premium: It is valued higher than the discount factor. This result is known to emerge in models with idiosyncratic shocks (see, for example, Krusell et al. (2011)), but it highlights that the result has nothing to do with risk aversion and much less with the third derivative of the utility function.

The second claim is that relaxing the borrowing constraint decreases the price of the asset. A credit crunch that lowers k induces a negative real interest rate. Hugget (1993) obtains a similar result numerically in a Bewley-type model. $\sigma/(\sigma-1)$ is the mean of the Pareto distribution, so as σ increases, the mean decreases and so does q^*.[7] In the limiting case when $\sigma \to \infty$, the Pareto distribution converges to the Dirac delta function $\delta(\theta-1)$. In such a case, from (3.11a), $q^* \to \beta$. This is a representative-agent result.

The third claim is related to the second; note that for low values of σ, $\beta\sigma/(\sigma-1)$ may well be greater than one. Hence a tight borrowing constraint pushes q^* above unity, a *negative* real interest rate.

The fourth claim simply states that if the borrowing constraint is relaxed significantly, the equilibrium may break down as there is no invariant distribution of wealth. This bound is given by $\overline{k}$ in (3.11b), at which point q^* reaches its minimum value.

The fifth claim states that a tightening of the borrowing constraint with a higher mean and higher variance of the distribution of urgencies to consume increases the equilibrium price of the asset. This result, coupled with Claim 3, shows that there is a relationship between the underlying shape of the distribution of shocks and the likelihood that a tightening of the borrowing constraint will produce a small or negative real interest rate.

The following results further characterize the equilibrium and can be found by direct differentiation:

$$\frac{\partial z}{\partial q} < 0, \frac{\partial F}{\partial z} > 0, \frac{\partial \ell}{\partial z} > 0. \tag{3.12}$$

where ℓ is defined as:

$$\ell = F/(1-F) = z^{\sigma} - 1, \tag{3.13}$$

the ratio of lenders to borrowers for given q.

Relaxing the borrowing constraint k decreases the price of the asset, increases the threshold value z, increases the fraction of lenders F and increases the ratio of lenders to borrowers. Most of the intuition about these results can be understood in a very simple fashion, by portraying the solutions in terms of a simple supply and demand analysis that I develop in the next subsection.

[7] For the Pareto distribution: $\frac{\partial \mathbb{E}(\theta)}{\partial \sigma} < 0,\quad \frac{\partial \mathbb{V}(\theta)}{\partial \sigma} < 0,$ for $\sigma > 2$ (please see footnote (6)). Also, according to this last result, as the variance of the distribution F increases (if it exists), so does q^*.

3.2. A simple interpretation

Market clearing admits a simple interpretation. The equality between supply of funds and demand of funds (2.3) can be written as:

$$
\begin{aligned}
S &\equiv \int_{\mathcal{A}} \frac{a+y}{q} d\Psi(a) F = \frac{\beta-(q-\beta)(\sigma-1)}{\beta q} \\
&= \frac{(q-\beta)(\sigma-1)}{\beta} k = b \int_{\mathcal{A}} d\Psi(a)(1-F) \\
&\equiv D
\end{aligned}
\tag{3.14}
$$

where I have used (3.9) and the assumption $b = ky$. From the definitions of supply of funds and demand of funds above:

$$
q = \beta\left[1+\frac{D}{(\sigma-1)k}\right], \quad q = \frac{\beta\sigma}{\beta S + (\sigma-1)} \tag{3.15}
$$

Figure 1 shows the demand and supply of funds as a function of the price q and the consequence of a credit crunch in which the parameter k shifts permanently from k to $k' < k$. As is evident, this change can push the economy to exhibit $q^* > 1$, a negative real interest rate.

The demand for funds D intersects the q axis at β, while the supply of funds S intersects the same axis at $\beta\sigma/(\sigma-1)$, which for low enough values of σ is higher than one. This opens the possibility of an equilibrium with negative interest rates. The demand for funds intersects the horizontal axis at $-(\sigma-1)k$. A credit crunch reduces k to k', the demand for funds rotates in the intersection at β, and the equilibrium q increases. Also, it is evident from the figure that the equilibrium price of the asset is always higher than β, which would correspond to the homogeneous-agent equilibrium result. The homogeneous-agent result for prices would arise as a limiting case when $\sigma \to \infty$, as explained previously. In the graphic, this would produce the degenerate case in which both curves S and D become flat at $q = \beta$, leaving the equilibrium quantities undetermined.[8]

When $k \to \infty$, we can see that the demand for funds D would be flat at the level β, graphically yielding this value as an equilibrium price with maximal borrowing. However, this point is infeasible; in fact, it violates (3.11b) which guarantees that the average asset holding is zero, that is, that condition (3.8) is satisfied.[9]

Note how the movement between equilibria from q^* to $q^{*'}$ involves both intensive and extensive margin adjustments. For example, each of the individuals supplying funds decreases the amount supplied to the market with an increase in price q. Also, since the increase in price decreases z as can be deduced from

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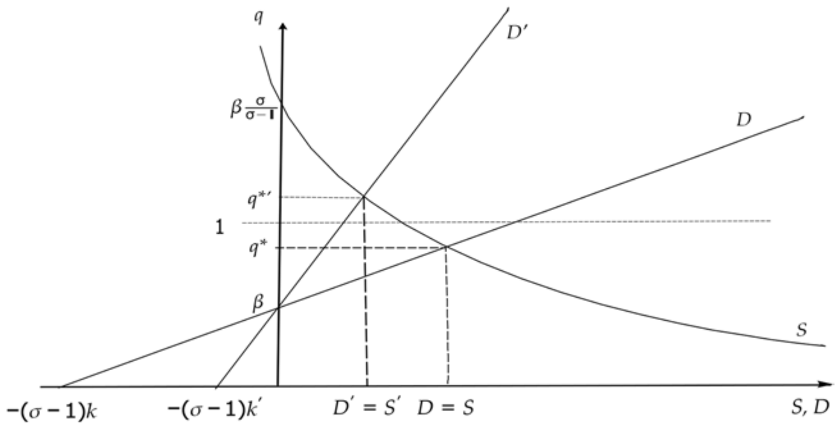
8 That both curves would be flat at β as $k \to \infty$ is straightforward for D in (3.15) and by applying L'Hopital's rule to S.

9 This limit could be introduced into the graphic by requiring that the negative values in the horizontal axis be smaller than $\overline{k}$ as defined in (3.11b).

(3.12), we know that the fraction of lenders F in equilibrium is reduced in favor of borrowers. Hence in the new equilibrium there are more borrowers and fewer lenders; the intuition of this is illustrated in Figure 1. A tightening of the borrowing constraint reduces demand for funds initially, which produces excess supply of funds. The excess supply of funds lowers the interest rate, an increase in q. This produces a change in the policy functions of individuals, (3.2), in which given that is more costly for lenders to save a given amount of resources, the threshold value z is lowered. Lenders reduce the set of possible values of θ such that they define urgencies to consume as small. On average, this creates a higher fraction of "desperate" individuals who become borrowers, reducing the fraction ℓ.

FIGURE 1

DEMAND AND SUPPLY OF FUNDS AND A CREDIT CRUNCH: INITIAL EQUILIBRIUM WITH q*. THE TIGHTENING OF THE BORROWING CONSTRAINT TO K' REDUCES DEMAND FOR FUNDS, INCREASING THE EQUILIBRIUM PRICE OF THE ASSET TO q*', ABOVE ONE



I now turn to wealth inequality and the influence of the borrowing constraint.

3.3. Wealth inequality

In this section I explore the influence of k on wealth inequality by using the Gini coefficient.[10]

[10] This result follows from the fact that for a random variable with distribution $\Psi(a)$, the Gini coefficient is defined as:

$$\mathcal{G} = \frac{1}{\mathbb{E}a}\int_0^\infty \Psi(a)(1-\Psi(a))\,da. \tag{3.16}$$

when $\mathbb{E}a > 0$. In the present case the mean of the distribution is zero, but this does not pose a problem because if each agent in the distribution is given the same quantity b,

$$\mathcal{G} = \frac{1}{b}\sum_{i=1}^{\infty}\Psi(a_i)(1-\Psi(a_i)) = \frac{1}{b}\sum_{i=1}^{\infty}[1-F(z)^i]F(z)^i = \frac{1}{b}\frac{F(z)}{1-F(z)^2}, \tag{3.17}$$

using equation (3.4).

Direct differentiation gives how the Gini coefficient changes with k:

$$\frac{\partial\mathcal{G}}{\partial k} = \frac{1}{b}\frac{\partial F}{\partial z}\frac{\partial z}{\partial q}\frac{\partial q}{\partial k}\frac{1}{1-F(z)^2} > 0, \tag{3.18}$$

where the expression is evaluated in q^*. The sign can be read off using (3.12).

The increase in k makes the lower end of the support of $\Psi(a)$ smaller, that is, more negative. Also, since the equilibrium price of the asset decreases, the policy function for lenders implies that the contiguous points in the support of the distribution become increasingly separated from each other; see equation (A.7) in the appendix. The direct impact of this is the spreading out of the support of the distribution. Counteracting this effect is the increase in $F(z)$, so the mass of the poorest individuals at $a_1 = -b$ who are $1-F(z)$ in number actually decreases. The net effect, according to the Gini measure, is to increase wealth inequality.

4. Conclusions

This paper develops a simple model that is useful for analyzing in detail certain questions regarding the influence of borrowing constraints in an economy where there are equilibrium credit arrangements. The model is sufficiently simple to deliver closed-form solutions for many objects of interest, in particular the distribution of wealth and policy functions. Many features of the model actually can be understood in a simple diagram of demand and supply of funds with analytical functions.

Distinct characterizations are obtained for the influence of the borrowing constraint on the equilibrium asset price, the composition of the credit market structure in terms of borrowers and lenders, and wealth inequality. The key assumptions underlying the analytical results are the linearity of utility in consumption and Pareto-distributed urgencies to consume. While the Pareto distribution can be justified on empirical grounds as it has been in related literature, the assumption of linear utility is harder to defend on the same basis. However, dropping this assumption would require resorting to numerical approximations in which it would not be possible to derive the effects of borrowing constraints with the level of detail presented here.

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the Gini coefficient does not change and the resulting mean is b, which has been used in (3.17).

References

Achdou Y., Han J., Lasry J., Lions P., and Moll B. (2014). "Heterogeneous Agent Models in Continuous Time". *Working paper*. Princeton University.

Aiyagari R. (1994). "Uninsured Idiosyncratic Risk and Aggregate Savings", *The Quarterly Journal of Economics*, Vol. 109; 659-684.

Atkenson A. and Lucas R. (1992). "On Efficient Distribution with Private Information", *The Review of Economic Studies*, Vol. 59; 427-453.

Bewley T. (1977). "The Permanent Income Hypothesis: A Theoretical Formulation", *Journal of Economic Theory*, Vol. 16; 252-292.

Farhi E. and Werning I. (2007). "Inequality and Social Discouting", *Journal of Political Economy*, Vol. 115; 356-402.

Hugget M. (1993). "The Risk-free rate in Heterogeneous-agent Incomplete-insurance Economies", *Journal of Economic Dynamics and Control*, Vol. 17; 953-969.

Krusell P. and Smith A. (1998). "Income and Wealth Heterogeneity in the Macroeconomy", *Journal of Political Economy*, Vol. 106; 867-896.

Krusell P., Mukoyama T. and Smith A. (2011). "Asset Prices in a Hugget Economy", *Journal of Economic Theory*, Vol. 146; 812-844.

Lucas R. (1980). "Equilibrium in a Pure Currency Economy", *Economic Inquiry*, Vol. 18; 203-220.

Lucas R. (1992). "On Efficiency and Distribution", *The Economic Journal*, Vol. 102; 233-247.

Taub B. (1988). "Efficiency in a Pure Currency Economy with Inflation", *Economic Inquiry*, Vol. 26; 567-583.

Taub B. (1994). "Currency and Credit are Equivalent Mechanisms", *International Economic Review*, Vol. 35; 921-956.

Wen Y. (2015). "Money, Liquidity and Welfare", *European Economic Review*, Vol. 76; 1-24.

Appendix

Proposition 1.

Proof. I use a guess-and-verify method to find the policy and value functions and assume that $v(a,\theta) = A(\theta) + B(\theta)a$. The Lagrangian for the problem (2.1) is:

(A.1)
$$\mathcal{L} = \max_{a'} \left[\theta(a + y - qa') + \lambda\left(\frac{a+y}{q} - a' \right) + \mu(a' + b) + \beta\mathbb{E}A(\theta') + \beta\mathbb{E}B(\theta')a' \right],$$

where λ and μ are the multipliers for the non-negativity constraint on consumption and the borrowing constraint, respectively. $\mathbb{E}$ is the expectation operator associated with $F(\theta)$. The first order condition is:

(A.2)
$$-q\theta - \lambda + \mu + \beta\mathbb{E}B(\theta') = 0.$$

I define z as the value of θ such that the agent with state (a, θ) would be indifferent between consuming and saving, in which case $\lambda = \mu = 0$:

(A.3)
$$z = \beta\mathbb{E}B(\theta') / q.$$

Given linearity of utility, it is natural to conjecture that agents who face $\theta > z$ would use all their resources plus maximal borrowing to consume while agents who face $\theta \leq z$ would not consume and save all of their income; this conjecture is reflected in policies (3.2). With the policies thus defined, the value function is expressed as:

(A.4)
$$A(\theta) + B(\theta)a = \begin{cases} 0 + \beta\mathbb{E}A(\theta') + \beta\mathbb{E}B(\theta')\dfrac{a+y}{q} & \text{if } \theta \leq z \\ \theta(a + y + qb) + \beta\mathbb{E}A(\theta') - \beta\mathbb{E}B(\theta')b & \text{if } \theta > z \end{cases}.$$

Equating coefficients:

(A.5)
$$A(\theta) = \begin{cases} \beta\mathbb{E}A\left(\theta'\right) + \beta\dfrac{y}{q}\mathbb{E}B\left(\theta'\right) & \text{if } \theta \leq z \\ \theta\left(y + qb\right) + \beta\mathbb{E}A\left(\theta'\right) - \beta\mathbb{E}B\left(\theta'\right)b & \text{if } \theta > z \end{cases},$$

$$B(\theta) = \begin{cases} \beta\dfrac{1}{q}\mathbb{E}B(\theta') & \text{if } \theta \leq z \\ \theta & \text{if } \theta > z \end{cases}.$$

Taking expectations:

(A.6a)
$$\mathbb{E}A(\theta) = \beta\mathbb{E}A(\theta') + (y+qb)\int_z^{\bar{\theta}} \theta\, dF + \beta\frac{y}{q}\mathbb{E}B(\theta')F(z)$$
$$-\beta b\mathbb{E}B(\theta')(1-F(z))$$

(A.6b)
$$\mathbb{E}B(\theta) = \beta\frac{1}{q}\mathbb{E}B(\theta')F(z) + \int_z^{\bar{\theta}} \theta\, dF$$

The i.i.d. assumption of shocks makes (A.6) a system of two equations in the unknowns $\mathbb{E}A(\theta)$ and $\mathbb{E}B(\theta)$, the solution to which gives all elements that make up Proposition 1. □

Proposition 2.

Proof. This proof consists of several steps. First I show that the support of the stationary distribution is countably infinite.

Lemma 1. $\Psi(a)$ has a discrete, countable infinite support, given by (3.6).

Proof. Assume that under fixed q individuals are "initialized" arbitrarily along $[-b,+\infty)$ in $\mathcal{A}$; eventually each individual would attain $-b$ and stay there as long as the urgency she faces is higher than z. When facing a draw $\theta \leq z$, she would save all her income and continue in this fashion as long as urgencies to consume are low. It follows that all agents will hold equity only in the states defined by the following recursion:

(A.7)
$$a_{i+1} = \frac{y+a_i}{q}, \quad i = 2,3,4..., \quad a_1 = -b.$$

This difference equation has a unique solution given by (3.6). Note how in (3.6), the support is well-defined for all values of q, even q=1, since in this case L'Hopital's rule applied to the second term implies that $\lim_{q\to 1}\frac{1-q^{1-i}}{q-1} = i-1$. □

Lemma 2. The law of motion for the distribution of agents by equity follows:

(A.8)
$$\Psi\left(a_{i+1}\right) = F(z)\Psi\left(a_i\right) + 1 - F(z)$$

where a_i is given by solution (3.6).

Proof. By (2.2), stationarity of the measure of agents requires:

(A.9)
$$\Psi(a') = F(z)\Psi\left(qa' - y\right) + 1 - F(z),$$

using policy functions for agents expressed in (3.2). Recursion (A.7) implies:

$$\Psi(a_{i+1}) = F(z)\Psi\left(qa_{i+1} - y\right) + 1 - F(z) = F(z)\Psi\left(a_i\right) + 1 - F(z). \tag{A.10}$$

Finally, recursion (A.10) is a difference equation with boundary initial condition $\Psi(a_1) = 1 - F(z)$, whose solution is (3.4). □

Proposition 3

Proof. Aggregate holding of the asset is computed from (3.7) as:

$$-\left(b + \frac{y}{q-1}\right)\frac{[1-F(z)]q}{q-F(z)} + \frac{y}{q-1} = 0. \tag{A.11}$$

We can see that a solution $q = 1$ is ruled out as an equilibrium outcome. The above equation can be rearranged as $[(1-F)kq - F](q-1) = 0$, where the assumption that $b = ky$ is used. I omit the argument of F for notational simplicity. Using the Pareto assumption of Assumption 1 and the value (3.10), it is possible to derive the following quadratic equation:

$$\Phi(q) = kq^2 + (1-\beta k)q - \beta\frac{\sigma}{\sigma-1} = 0. \tag{A.12}$$

The solution is given by (3.11a). By directly finding $F(q^*)$, it is straightforward after some algebra to show that the condition $F(z(q^*))/q^* < 1$ that bounds average holdings of the asset translates to:

$$\frac{4\beta^2 k\sigma}{\sigma-1}\left(k + \beta k\sigma - \beta k - k\sigma + \beta + \sigma - 1\right) > 0 \tag{A.13}$$

from which condition (3.11b) is derived. □

Claim 1

Proof. Most of the results follow from very simple algebra:

1. $q^* > \beta$ follows since the term in brackets in (3.11a) is greater than one, given that $\sigma > 1$.
2. $\frac{\partial q^*}{\partial k} < 0$, $\frac{\partial q^*}{\partial \sigma} < 0$ follows by direct differentiation.
3. $k \to 0$, $q^* \to \frac{\beta\sigma}{\sigma-1}$ can be shown by directly applying that limit to q^* in (3.11a) and using L'Hopital's rule.

4. That the minimum feasible value of q^* is attained when $k \to \bar{k}$ follows because q^* is strictly decreasing in *k*.
5. $\frac{\partial^2 q^*}{\partial k \, \partial \sigma} > 0$ can be shown by computing the derivative, the sign of which is the same as the sign of the term $\beta k - 1 + 2\frac{\sigma}{\sigma - 1}$. Even when $k = 0$ this term is positive since $\frac{\sigma}{\sigma - 1} > 0$, hence the result follows.