The quantum harmonic oscillator expected shortfall model*

El modelo de déficit esperado basado en el oscilador armónico cuántico

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Abstract

This paper presents a new Expected Shortfall (ES) model based on the Quantum Harmonic Oscillator (QHO). It is used to estimate market risk in banks and other financial institutions according to Basel III standard. Predictions of the model agree with the empirical data which displays deviations from normality. Using backtesting, it is shown that the model can be reliably used to assess market risk.

Key words: *Expected Shortfall; market risk; Basel III standard; stock returns; S&P index.*

JEL Classification: *G24, C22, C52, C53.*

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Resumen

Este documento presenta un nuevo modelo de déficit esperado basado en el oscilador armónico cuántico para la estimación de riesgo de bancos e instituciones financieras conforme al estándar de Basilea III. Las predicciones del modelo son consitentes con los datos del mercado accionario que presentan desvíos de normalidad. Utilizando "backtesting", se muestral que el el modelo es fiable para la evaluación del riesgo de mercado.

Palabras clave: *Déficit esperado; riesgo de mercado; Basilea III; retorno accionario; S&P.*

Clasificación JEL: *G24, C22, C52, C53.*

1. Introduction

Back in the 1960s, Mandelbrot (1963, 1972) and Fama (1965) showed that the series of daily returns of securities have a distribution that deviates from the normal distribution and from the identical and independent distribution assumption. Fama (1965) assumed that the distribution of price change is approximately Gaussian or normal, which was confirmed by observations. It was found that extreme tails of empirical distributions are higher than those of normal distribution, and four parameter Paretian distribution was introduced to describe data. Blattnerg and Gonedes (1977) showed that returns distributions are characterized by fat tails. They considered another family of symmetric distributions that can consider fat tails. It was Student or t distribution, and authors concluded that Student model has greater descriptive validity then the normal distribution. Kan and Zhou (2017) also presented similar findings using multivariate t distribution with 7 degrees of freedom to model stock returns. They point out that due to the presence of fat tails, the assumption of normality must be rejected. Empirical evidence of non-Gaussian properties of stock market return distribution led to the development of a lot of theoretical models on this subject. From the econophysics point of view, the Brownian movement of the classical particles was used to model the stock returns in the first place (Dragulescu and Yakovenko, 2002; Roumen, 2013; Reddy and Clinton, 2016; Agustini *et al*. 2018). Change of the stock price return is modelled as position change of random displacement of classical Brownian particle in these papers. The main problem with this model is that lead to Gaussian-type processes (Madan and Seneta 1990). Traditional economic models were developed to better describe the stock return distributions (Linden, 2001; Dragulescu and Yakovenko, 2002). On the other side, the real data and empirical stock return distributions show deviations from Gaussian type distributions since Probability Density Function - PDF tails decay slower than log-normal Gaussian type (Şener *et. al*. 2012; Zikovic and Filer, 2013; Rossignolo, *et. al*. 2012, 2013, Radivojevic *et al*. 2016b, 2017a, 2020; Doncic *et al*. 2022). Fat tails which include negative skewness on one side and positive excess kurtosis on the other side of the center of distribution are the most common types of deviations from Gaussian type distribution (Ahn *et al*. 2017).

In the market models based on statistical physics, which try to make the analogy of the stock market behavior with microsystems in physics, an important role found quantum mechanics (QM), which naturally inherent statistical fluctuations via uncertainty principle (Ataullah *et al*. 2009). The main problem in QM is that the potential that describes the interaction of the physical system (which is used to describe market) with the environment is generally unknown. To use QM models to describe the stock market return distributions, the appropriate potential is needed (Zhang and Huang, 2010; Haijun and Guobiao, 2015; Wróblewski, 2017). The main principle is to make an analogy between some QM system, e.g. quantum particle (or systems of particles) and stock price return. In Schrodinger's nonrelativistic QM of closed systems, the particle is described with wave functions of particle state. Physically meaning has a square of the amplitude of wave function, which should describe the PDF of stock market returns. This is the merging point of stock market returns and the QM system: there is a need for QM system with wave function, which square can describe PDF of stock market returns. Closed quantum systems with time independent potentials lead to stationary states, so some perturbation potential needs to be introduced to enable time evolution and nonstationary.

It is interesting to note that stock return distributions of stable markets tend to have Gaussian properties. In general, all markets tend to reach an equilibrium state (Balvers *et al*. 2000), and settle to some form of Gaussian-like distributions shape (Ahn *et al*. 2017). Fat tails are one of the most common deviations. Stock market returns tend to settle in some equilibrium or near-equilibrium state, which can be described as a true or local minimum of the potential energy in the physics analogy. A market can be described as some sort of physical system which is in equilibrium or near equilibrium with its surrounding. Quantum mechanical systems which are isolated can be described with the Schrodinger equation in which the parameter that need to be known is its potential energy or potential. Since the potential is unknown, some reasonable guesses need to be introduced and substantiated with some real physical assumptions (Zhang and Huang, 2010). Stock markets returns in general tend to long-run equilibrium, where returns dissipate around some mean value. It implies that a

QHO can be used to describe these oscillations, which fluctuate over time, so first order perturbation theory needs to be introduced. Hence, the aim of this study is to take advantage of this opportunity.

Among the first was Bachelier (1900), who described the financial assets price movement using a random walk model, and introduced the concept of Brownian motion, which is a type of random process that has played a fundamental role in the development of modern mathematical finance. From the point of view of the current paper, random processes in economics can be transformed into the form of Schrodinger equation (Wroblewski, 2017; Ahn *et al*, 2017; Vukovic *et al*, 2015), which is a fundamental equation in Quantum physics. For instance, the famous Black-Scholes equation which gives a model to pricing theory is an instance of Schrodinger equation (Vukovic et al, 2015; Contreras et al, 2010). It was shown by Vukovic et al, 2015, that starting from Black Scholes equation, using mathematical transformations, one can get to the exact form of Schrodinger equation. Phenomena that have the same or similar mathematical foundations in different disciplines, will have same or similar physical behavior.

Important property of QHO is that like every bounded quantum system it has eigenstates and discrete spectrum of energies. Hence QM oscillator can be described with one of eigenstates or superposition of eigenstates. This practically means that QHO can be described as linear combinations of eigenstates. Like classical Brownian particle, QHO in ground state is described with Gaussian distribution. Since stock markets show deviations from Gaussian (negative skewness and positive excess kurtosis) classical Brownian particle is not quite suitable for describing it. On the other hand these deviations can be very well described with higher states of QHO. Eigenstates of QHO are Hermitian polynomials, which can be even or odd. Even states lead to more symmetric distributions and can contribute to the fat tail and lead to higher kurtosis. Odd states lead to distributions with a larger skewness, (Ahn *et al*, 2017).

The paper is organized as follows: Section 1 contains the introduction. The following section gives an overview of the most significant empirical research in the area of ES models. Section 3 presents the theoretical basis of the possibility of applying QHO for predicting the movement of stock market returns. In Section 4 presented results of applying QHO. In Section 5, the backtesting results are presented, analyzed, and discussed. Section 6 summarizes the conclusions.

2. Literature Review

There is an abundance of papers in literature dealing with the improvements of the applicability of different market risk models according to Basel Commitment rules. All those papers can be classified into two groups. The first group consists of the papers which try to improve applicability of different ES models. In this group of papers researchers use a traditional technique for predicting behavior patterns of assets in financial markets, following known distributions. Such papers were presented by Barone-Adesi and Giannopoulos (2001), Pascual *et al*. (2006), Chen *et al*. (2011), Brandolini and Colucci (2012), (2012), Alemany et al (2012), Bee (2012), Radivojevic *et al*. (2016, 2017, 2020) etc. The second group includes the papers which try to improve the applicability of completely different models for prediction stock returns. Those papers are based on artificial intelligence, data mining, machine learning, and similar concepts for assessing risks to which participants in financial markets are exposed. Such papers were presented by Scaillet (2003 and 2004), Fermanian and Scaillet (2005), Atsalakis and Valavanis (2009), Thomaidis and Dounias (2012), Aguilar-Rivera et al. (2015), Cavalcante *et al*. (2016), Chong et al. (2017) Xing *et al*. (2018), Hiransha *et al*. (2018), Fischer and Krauss (2018), Rundo *et al*. (2019), Nti *et al*. (2019), Shah *et al*. (2019), Sezer *et al*. (2020), Doncic *et al*. (2022) etc.

From the second group of papers, one can single out papers that focus on opportunities of applying solutions from physics. Such papers were presented by Meng *et al*. (2016), Agustini *et al*. (2018), Maruddani and Trimono (2018) etc. Inspired by a series of studies that successfully used concepts and tools from quantum mechanics to options pricing (Ye and Huang, 2008; Baaquie, 2009; Bagarello, 2009; Zhang and Huang, 2010; Pedram, 2012 and Cotfas, 2013), Agustini et al. (2018) were use Geometric Brownian Motion model for stock prices prediction. Like them, Maruddani and Trimono (2018) used multidimensional Geometric Brownian Motion model to describe stochastic process of stock price movements. However, despite of the mathematical success of quantum-mechanics models for financial instruments, only few studies have been tried to exploit quantum statistical dynamics relying on open-system concepts yet (Meng et al. 2016). The justification for applying solutions from QM can be found in empirical findings that point to the unsustainability of the efficient market hypothesis. Empirical findings such as non-Markovian memory (Wan and Zhang, 2008) and fat-tail deviation (Wan and Zhang, 2008, Radivojevic *et al*. 2020) suggest that the stock market does not satisfy the classical Brownian motion model (Ye and Huang, 2008). And Meng, *et al*. (2015) were among the first to point out the possibility of describing dynamical problems in the stock market using a wave function. In this context, Meng *et al*. (2016) were among the first authors who presented the idea of the possibility of applying the Brownian motion quantum oscillator model. They showed that the movement of financial asset returns can be described by the Markovian Klein-Kramers equation. However, they focused only on predicting the movement of stock prices, without considering the possibility of applying the model for assessing market risk. Original idea from QM, that the more we learn of the coordinate the less we know the momentum (and vice versa) (Cohen-Tannoudji *et al*. 1992), can be applied to stocks because the more we know the stock price the less information we can use to estimate the trend of it (Meng *et al*. 2016).

In QM this is the Heisenberg uncertainty principle, which states that one cannot with certainty know position of particle and its momentum or speed. This principle has its analogy in the economy. As Ye *et al*, 2008 stated if all the people know the price value, even though the price has deviated, it will turn back to the value swiftly and never start to fluctuate again. In this sense, precise knowledge of stock price will harden estimates of its change (see Ye et al, 2008 for details).

3. The Theoretical Basis Of The Possibility Of Applying QHO For Predicting The Movement Of Stock Market Returnsw

In case of harmonic potential QM solutions of the stationary Schrodinger equation are already known and are

(1) $$\psi_n(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right) e^{-\frac{m\omega}{2\hbar}x^2},$$

where $H_n\left(\sqrt{\frac{m\omega}{\hbar}}x\right)$ are Hermite polynomials. Expression in equation 1 can be simplified by introducing dimensionless variable $\xi = \sqrt{\frac{m\omega}{\hbar}}x$. In physics, x is the coordinate of the observed particle. This physical quantity has its analogy in the stock market model $x \equiv \ln s$, which is logarithmic stock price return, where s is stock price (Meng *et al*, 2016). Parameter m is the mass, while ω is the angular velocity of the quantum particle, $\hbar = 1.05457 J \cdot s$ is reduced Planck constant and $n = 0,1,2,...$ is positive integer number. Angular velocity is related to the energy of the quantum particle as

(2) $$E_n = \left(n + \frac{1}{2}\right)\hbar\omega.$$

According to Ye and Huang, 2008 and Meng et al 2015 and 2016 mass of particle correspond to inertia of the stock, energy of the particle corresponds to trading volume of the stock, while $\psi_n(x)$, which is called wave function, has special properties. As it was stated earlier in the text, the wave function doesn't have physical meaning in physics, but its square of amplitude represents the probability density of finding a particle with coordinate *x*. Square of amplitude of the particle corresponds to the probability density distribution of the stock price (Ye and Huang, 2008 and Meng *et al* 2015; Meng *et al* 2016).

General solution of stationary Schrodinger equation for the potential of the harmonic oscillator can be constructed in form of infinite expansion over Hermite polynomials or to be more precise over eigenstate wave functions given by equation 1:

(3) $$\psi(x)=\sum_{n=0}^{\infty}c_n\psi_n(x).$$

Expansion coefficients have important physical implications: its square gives the probability that the system can be found *n*-th state.

Since stock returns change over time, we need to introduce some small potential, which acts as a perturbation on our quantum system and changes states over time. If the perturbation is small, we could expect that eigen states are not changed due to the influence of small potential, and in first-order perturbation theory we could expect that the state of the system can be described as

(4) $$\Psi(x,t)=\sum_{n=1}^{k}c_n(t)\psi_n(x)e^{-i\frac{E_n}{\hbar}t}.$$

Perturbation leaves eigen states unchanged, but consequently, expansion coefficients are time dependent.

Let us assume that stock return distribution can be described as the probability distribution function of the QHO. Many authors use QHO to model stock return distributions (Tingting and Yu, 2017; Jaroonchokanan and Suwannay, 2018). Since the market has a tendency toward an equilibrium state (Menga *et al*. 2016), with some amount of fluctuations, it is quite reasonable to assume that model of QHO is mostly in the ground state, which has Gaussian shape properties, with additional impurities of excited states, which lead to the fat tails and non-Gaussian properties. Some form of the superposition state could be a real representation of the market model (Menga *et al*. 2016). General state function can be represented as

(5) $$\psi(x)=\sum_{n=1}^{k}c_n(t)\psi_n(x),$$

where expansion is limited to some k-th excited state.

Let us further build our model. We can take stock returns of some real markets for three years period, represented by 750 daily stock price returns. Our goal is to use the first two-year period to build the QHO model and predict stock returns of the third year.

4. Model Estimation

At start, we take the first year of stock returns to construct initial PDE of the real market. This function needs to be fitted with the square of the QHO superposition state function in the form

(6) $$\psi(x) = \sum_{n=0}^{9} c_n(0)\psi_n(x)$$

In the expansion series, let us assume the first ten members (greater members are neglectable small, which will be seen in result part of the paper).

It is important to notice that the PDE of the stock return should be fitted with the square of the function in explicit form

(7) $$\psi(\xi) = \sum_{n=0}^{9} c_n(0)\left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_n(\xi) e^{-\xi^2/2}$$

In our QHO model all parameters are uncertain, i.e. mass m and angular frequency ω.

These parameters should be extracted from the data of the real market. To do so, let us first introduce function

(8) $$f(x) = \left(\sum_{n=0}^{9} a_n H_n(x)\right)^2 e^{-\kappa^2 x^2}$$

which will be used to fit stock market PDE. This is necessary, since it is important to find an appropriate function which can have enough degrees of freedom, so the iteration procedure of finding fitting coefficients can lead to minimal residuals. Finding the best fit is procedure to find ten coefficients and an additional κ unknown coefficient which is in correlation with all ai coefficients. Linear regression procedure of finding best parameters starting from best guess initial values of coefficients, lead to great accuracy of fit. Here, the fitting procedure was done not over the variable x, but instead, independent variables are Hermite polynomials $H_n(x)$. This complicates the procedure a bit since function $f(x)$ need to be minimized over terms of Hermite polynomials.

To perform fit over Hermite polynomials as independent variables, a leastsquare fit method was used. This method is based on minimizing χ^2 function, or minimising expression $\sum r_i^2$, where r_i are residual, or differences between original data point and its fitted value. In order to perform a such fit, initial values of the fitting parameters need to be set. The parameters can be arbitrary, or based on the intuitive knowledge of the fitting curve. Here, we can make assumption that ground level of the QHO is dominant and have contribution about 90% (initial guess), which will give expansion coefficient of the ground level $\sqrt{0.9} \approx 0.95$. Excited states of the QHO are abundant with less probability, and expansion coefficients in equation 3 to 7 need to fulfill constraint $\sum c_i^2 = 1$. Mathematica Wolfram provide fitting algorithm (Wolfram, 2022) used to perform fitting over Hermite polynomials in the above described way. Fitting procedure is based on minimising χ^2 function over paremeters c_i.

Afterwards, it is necessary to relate a_n fit coefficients in eq(8) with c_n coefficients in eq (7). To do so, in function $f(x)$, Hermite polynomials need to be factored, in form $H_n(\kappa x)$ due to correspondence with eq(7). $\kappa = \sqrt{\frac{m\omega}{\hbar}}$ is determined from fit, and κ can be trivially factored in form $H_n(\kappa^{-1}\kappa x)$. If we introduce $\gamma = \kappa^{-1}$, we can transform term $H_n(\gamma\kappa x)$ to $a_n H_n(x)$ as following

$$\begin{aligned}&\sum_{n=0}^{9} a_n H_n(x) = \\ &\sum_{n=0}^{9} a_n \sum_{i=0}^{Floor[n/2]} \gamma^{n-2i} (\gamma-1)^i \binom{n}{2i} \frac{(2i)!}{i!} H_{n-2i}(\kappa x) = \\ &\sum_{n=0}^{9} A_n H_n(\kappa x)\end{aligned} \quad . \tag{9}$$

Now we can expand right side of equation (9) and collect coefficients A_n beside $H_n(\kappa x)$ members. Finally, our state expansion coefficients are

$$c_n = A_n \left(\frac{\xi^2}{\pi}\right)^{-\frac{1}{4}} \sqrt{2^n n!} \, . \tag{10}$$

Now our state can be expressed in terms of known coefficients

$$\psi(x) = \sum_{n=0}^{9} c_n \left(\frac{\kappa^2}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\kappa x) e^{-\frac{\kappa^2 x^2}{2}} \, . \tag{11}$$

Following the above procedure, PDF from stock return empirical data can be created, and fitted with Hermite polynomials.

To build a predictive stock return model, the first year period, i.e. the first 250 data points from the stock return values, are used to generate PDF which represents the initial state of our QHO. The natural states of the real model are not stationary and are changing over time. To explain non-stationarity, perturbation is introduced, resulting in the time-dependent wave function given by eq(4). To obtain time dependence of expansion coefficients in eq(4) second year period of stock return data is used. The initial state of QHO is created from 250 data from the first-year period, while the next state is created using data from the 2nd to the 251st day. In this way, the window consisted of 250 data points representing one state is created, and this window is moved forward for one day. By continuing procedure of fitting PDF with QHO, wave functions and time expansion of coefficients in eq(4) can be found, using the method of rolling window. At this point, it is necessary to state that the mass of the QHO is one parameter introduced in fitting equations. By finding the best fit, this parameter changes in rolling windows since it is not fixed. To fix the mass of the QHO, second-year data of stock return distributions were used and fitted using the rolling window method. Afterwards, the most probable mass is extracted as a parameter. Now fitting procedure must be repeated for all second-year stock return data to obtain new fitting coefficients, which correspond to the fixed mass of the QHO. At the end of this procedure, coefficient of expansion in eq(4) are known.

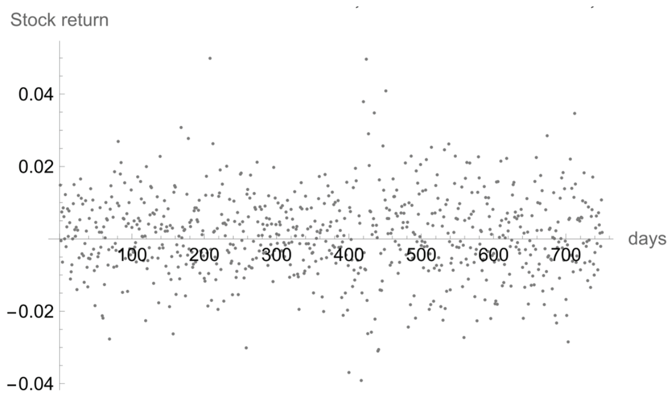
If we can assume that time evolution of expansion coefficients, $c_n(t)$, are due to some weak perturbation, it is reasonable to suggest that this evolution can be extrapolated to a future time. Uncertainty of prediction increases over time, but for us, it is important to find PDF for the following day, related to the present one. It is enough to use a two-year period (500 data points) to predict one day after two-year period (501 data point). For the following day (502 data point) we do not need to predict expected stock returns from the first 500 data points. Instead, we can use 501. data point and repeat procedure with building PDF, fitting empirical data and predict outcome for just next day. This repeating procedure can lead us through the whole third year, and predictions of the model can be compared with real data. It is important to note that this model is probabilistic, and by predicting time series coefficients, $c_n(t)$, for the following day, we are predicting tomorrows wave function, $\psi(x)$, of QHO. This information does not give us data values that will occur the following day, but rather PDF of the following day ($\text{PDF} = |\psi(\text{x})|^2$).

5. The Results Of QHO Model Application

To verify presented model, we need to start with real data of the stock return. These data are presented on the Fig 1 for the three years period. The data used were the daily logarithmic returns of the Standard and Poor stock index, which was used in Christoffersen (2011) for VaR estimates. The returns were collected for the period between January 2st, 1997 to December 30th, 1999. These empirical data are starting point of building model, following Methodology procedure.

FIGURE 1

STOCK RETURN VALUES FOR THREE YEAR PERIOD – 750 DATA POINTS FOR THE S&P INDEX FOR THE PERIOD BETWEEN JANUARY 2ND, 1997 TO DECEMBER 30TH, 1999



The first year of the three-year period is used to build a PDF of the stock return. In this function, all information about the behavior of the stock market is hidden and needs to be extracted. PDF is created by discretizing stock return values and counting the number of data that falls into the correspondent bin. To have proper distribution properties (i.e. Probability Density Function), distribution needs to be normalized to the unit. Connection with QHO is that PDF generated from real data gives probabilities of measuring QHO with a given x coordinate. On the other side, this PDF of QHO is equal to the square of the quantum wave function which describes a state of QHO. QHO is described with real functions and written in the general form given by eq (7) and can be used to fit empirical PDF. The result within the range of empirical data is shown in Fig 2 for one 250 days period. Figure 2 included a fit with Gausian and Pareto type IV function. Gaussian function describes the random walk of Brownian particles, while general Pareto distribution – GDP is used to approximate asymptotic distributions of extreme values.

From Figure 2 the QHO function can more realistically describe empirical data. Figure 3 shows residuals in the fit procedure for three mentioned functions. QHO fits data better compared to Gaussian and GPD distribution (Pearson N., 2002). One of the reasons is in the fact that QHO has greater degrees of freedom, and odd members in the expansion can consider skewness, while even one capture kurtosis. In this way, any arbitrary function can be fitted. If not with given degrees of freedom, then with more degrees of freedom, which can be easily added. Reason for this lies in the fact that the QHO function is a kind of expansion in series, but not over the variable, but over Hermite polynomials that depend on the variable.

FIGURE 2

DOTS-EMPIRICAL PDF OF STOCK RETURNS FOR ONE YEAR PERIOD; DASHED LINE – GAUSSIAN FIT; DOTTED LINE – PARETO TYPE IV DISTRIBUTION FIT; FULL LINE – FITTED FUNCTION OVER THE SUPERPOSITION STATE OF QHO

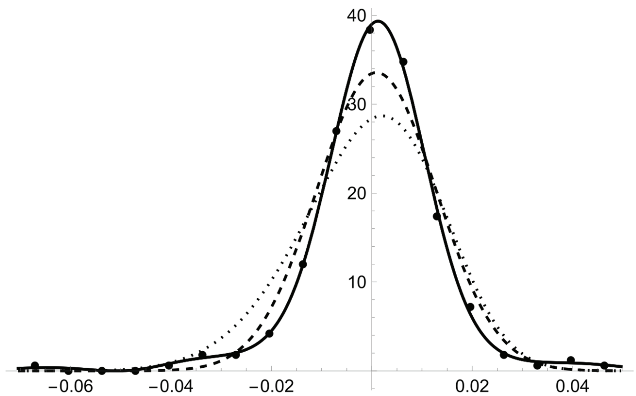
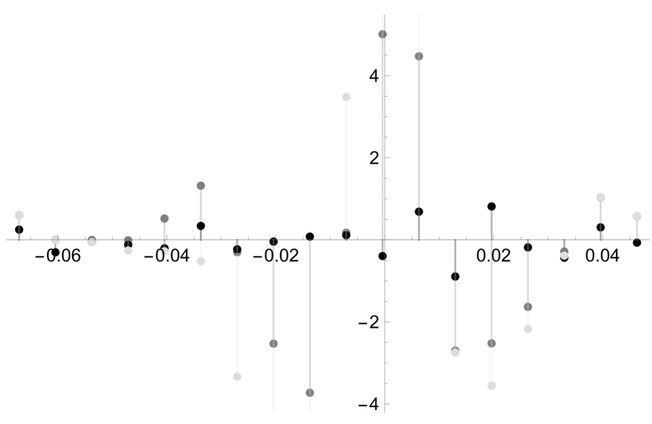


FIGURE 3

GRAY DOTS – RESIDUAL FROM GAUSSIAN FIT; LIGHT GRAY DOTS – RESIDUALS FROM PARETO FIT; BLACK DOTS – RESIDUALS FROM QHO FIT.



It is interesting to note that fitted QHO function does not diverge in the region out of the fitted data (fitting function is defined in the region $(-\infty, +\infty)$) and the norm of the fitted function given at Fig 2 is

$$\int_{-\infty}^{+\infty} |\psi(x)|^2 \, dx = 0.991002 \tag{12}$$

The function has a norm, which is less than one per cent smaller than the unit, which leads us to the conclusion that the superposition state of QHO is natural state of stock return. This is the case with all 250 PDF functions generated from a two-year period of stock returns. A number of iterations and precision goal of the fitting procedure are set in a way, that norm of all functions needs to be in the range of one per cent around the unit. Values of the coefficients of the fitted wave functions from Fig 2 – initial state, and their squares, are given in Table 1.

TABLE 1

FITTING COEFFICIENTS AND ITS SQUARES FOR THE EMPIRICAL PDF OF FIRST YEAR PERIOD OF STOCK RETURNS

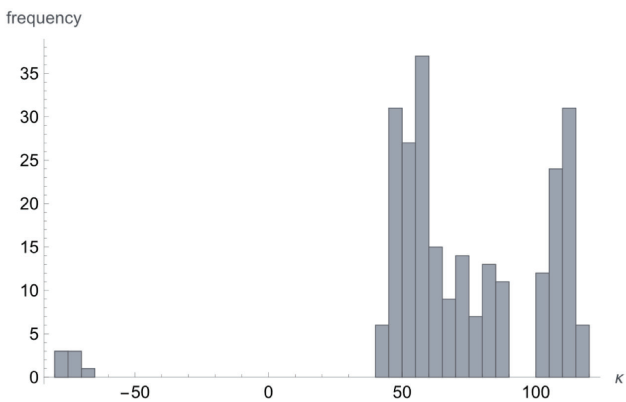
C_0	0.984902	C_0^2	0.970032
C_1	0.0603988	C_1^2	0.00364801
C_2	0.125271	C_2^2	0.0156929
C_3	-0.0135641	C_3^2	0.000183984
C_4	0.0268013	C_4^2	0.000718308
C_5	0.00142965	C_5^2	$2.0439 \cdot 10^{-6}$
C_6	0.0274061	C_6^2	0.000751093
C_7	0.0012143	C_7^2	$1.47453 \cdot 10^{-6}$
C_8	$-2.2201 \cdot 10^{-6}$	C_8^2	$4.92883 \cdot 10^{-12}$
C_9	$-8.63369 \cdot 10^{-8}$	C_9^2	$7.45406 \cdot 10^{-15}$

Looking at the values given in Table 1, QHO is in a superposition state, where the most probable is the ground state. Excited states give a smaller contribution to the overall wave function. This fact has physical implications. Ground state of QHO has Gaussian shape, and the tendency to Gaussian-like stock returns of the markets is conserved. Excited states are odd and even and each gives contributions to different properties of stock markets return (skewness and kurtosis). Going further to higher excited states will give even smaller contributions and can be neglected.

Following the fitting procedure of rolling window for the second year of stock returns wave functions which represent the state for every day are obtained. These functions still have no significant physical meaning, since they present only the mathematical best fit of the data. As the fitting parameter in every fit, the quantity $\kappa = \sqrt{\frac{m\omega}{\hbar}}$ of the oscillator arises, and it has different values in each fit iteration. For the harmonic oscillator, this value needs to be constant. Idea is to find the best fit in each iteration and then choose the most probable one. Figure 4 gives the histogram of the occurrence of the values of κ. The most probable value is 49.684 m^{-1}, and this value is fixed as the parameter of the oscillator. At the first look at the Fig. 4 some paradox fact arise - the negative value of parameter κ. One need to keep in mind that these values still have no physical meaning, rather than pure mathematical products of the criteria for the best fit. Another thing that can be noticed is that histogram has two modal properties. Again, this is pure mathematical property of accumulating possible values around the two values. The presented model does not include two coupled oscillators, nor the changeable mass of the oscillator. Hence, the most probable value has been chosen. Also, the occurrence of negative mass has no physical implications and needs to be rejected.

FIGURE 4

HISTOGRAM DISTRIBUTIONS OF OCURRENCE OF DIFFERENT VALUES OF CONSTANT κ



At this point with the fixed mass of the oscillator, the fitting procedure can be repeated from the beginning, while this time parameter κ is no more changeable. Finding best fits leads us to different values of expansion coefficients in eq. (6). For the second-year window of data, 250 values of expansion coefficients are obtained. Their values are presented in Figure 5.

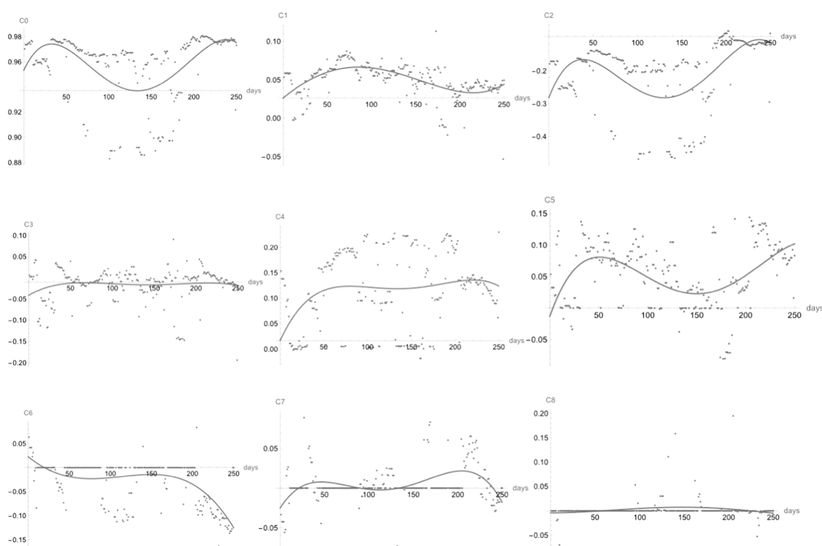
The perturbating potential is unknown, and possible form of time-dependent function that expansion coefficients should have been undetermined. This led us to use polynomial series expansion and assume that power series consisting of a few first members will be a good approximation of general function. Each expansion coefficient is now fitted with a time-dependent function in the form of

$$C_n(t) = A + Bt + Ct^2 + Dt^3 + Et^4 \tag{13}$$

Fitting functions for each expanding coefficient $C_n(t)$ are presented in Fig 5 for rolling windows of the second year of stock returns. These functions are then used to calculate the expanding coefficients off the first day of the third year period of the stock returns, which is at this point an unknown variable. This enables us to write down wave function for the unknown tomorrow data, determine PDF and predict tomorrow's outcome with a certain probability. Predictions can be extrapolated to a future period, greater than one day, but uncertainty can be large. Instead, we can wait for the empirical value for the next day and repeat the above procedure to obtain the wave function for the following day. Since empirical data in our model are known for a whole three-year period, we iteratively repeated the procedure to predict outcomes for the whole third year.

FIGURE 5

TIME EVOLUTION OF EXPANSION COEFFICIENTS FOR THE SECOND YEAR PERIOD OF STOCK RETURNS



To validate our model, predicted mean values of expected stock rerun with limits of one standard deviation are presented in Fig 6 and compared with empirical data. To calculate the standard deviation

(14) $$\sigma = \sqrt{\langle x^2 \rangle - \langle x^2 \rangle},$$

mean, $\langle x \rangle$ and mean square values, $\langle x^2 \rangle$ need to be calculated

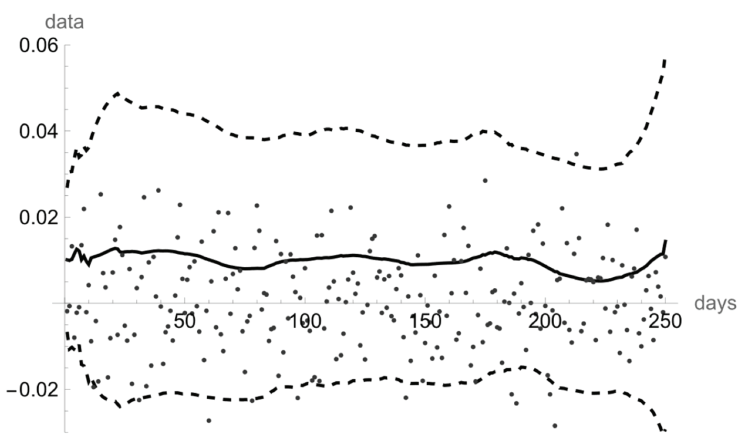
(15) $$\langle x \rangle = \int_{-\infty}^{+\infty} \psi^* x \psi \, dx \text{, and } \langle x^2 \rangle = \int_{-\infty}^{+\infty} \psi^* x^2 \psi \, dx,$$

where asterix refer to complex conjugation. These operations doesn't change real functions.

Since PDF distributions are roughly Gaussian shape distributions, the interval of one standard deviation covers approximately 68% of the whole range. It is expected that two-thirds of the whole data fall into the presented region in Fig 6. Newer the less only 4% of data is outside of the region. Sock return distributions for data used in the presented model have negative skewness and fat left tail and distributions are asymmetric around the mean value. This is the reason why more data are below the mean curve in Fig 6.

FIGURE 6

STOCK RETURN DATA IN THIRD YEAR – DOTS; PREDICTED MEAN – FULL LINE; STANDARD DEVIATION INTERVAL AROUND MEAN – DASHED LINE; FOR THE PERIOD BETWEEN JANUARY 2ND, 1997 TO DECEMBER 30TH, 1999



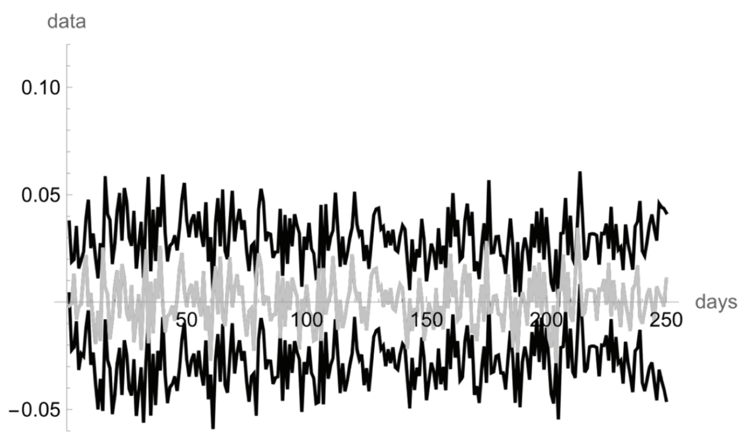
A powerful tool in Quantum Mechanics are selection rules that can be derived from wave functions, determine possible transitions between quantum states and give transition probabilities. This could limit the possible outcomes and better predict tomorrow's stock returns. For the derivation of selection rules, an exact form of perturbation potential is needed. At this point, the best-predicted value of tomorrow's outcome can be obtained using the traditional formula

$$P_{t+1} = P_t \pm \mu \cdot \sigma \tag{16}$$

where P_{t+1} is tomorrow's stock return, P_t is stock return at present day μ is the quantile that gives confidence interval (taken to be unit) and σ is the standard deviation calculated using the predicted wave function for tomorrow outcome. Eq (16) has great importance, since implies that tomorrow's outcome is directly related to today's stock return values. On Figure 7 are presented predicted outcomes given with full line, while empirical tomorrow's outcome is plotted with gray line. It is of great importance to notice that outcomes do not falls out of the predicted interval.

FIGURE 7

PREDICTED OUTCOME OF STOCK RETURN – FULL BLACK LINE; EMPIRICAL STOCK RETURN – FULL GRAY LINE; FOR THE PERIOD BETWEEN JANUARY 2ND, 1997 TO DECEMBER 30TH, 1999

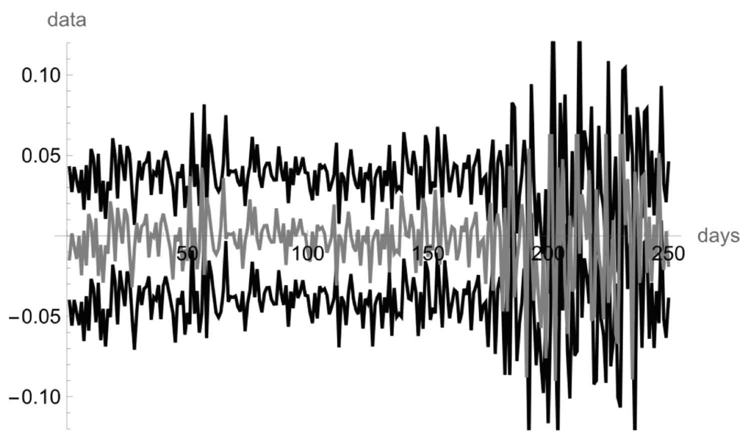


6. Backtesting Model According to Basel III Standards

The obtained data were used to estimate the Expected Shortfalls (ES) according to Basel III standards (Bank for International Settlements, 2013). More precisely, the ES are calculated for the one-day-ahead horizon for the period from January 1st, 1998, to January 1st, 1999, according to the Basel III standard, and for the period from January 1st, 2008 to January 1st, 2009, which was during an economic crisis, Figure 8.

FIGURE 8

PREDICTED OUTCOME OF STOCK RETURN – FULL BLACK LINE; EMPIRICAL STOCK RETURN – FULL GRAY LINE, FOR THE CRISIS PERIOD FROM JANUARY 1ST, 2008 TO JANUARY 1ST, 2009



The ES estimates were made for the confidence levels of 97.5%. Since VaR does not fulfil all the characteristics of coherent risk measures, the Basel Committee has proposed fundamental changes in the regulatory treatment of financial institutions' trading book positions (Kellner and Rösch, 2016). Among other things, the replacement of 99% VaR with the 97.5% expected shortfall (ES) for the quantification of market risk is recommended (Radivojevic *et al.*, 2019; Doncic *et al.*, 2022). The rest of the observations were used as the resample observations needed for the ES starting values. At the same time, to answer the question of whether the model contributes to the improvement of risk assessment, i.e., whether the model gives better results compared to traditional risk models, the performance of the model was compared with the performance of three ES models: GARCH models under the assumption that innovations follow the Student's T, GED and Skewness GED distributions. Models were chosen given in the mind results of studies conducted by Radivojevic *et al*. (2015) and Rossignolo *et al*. (2013, 2012). The maximum likelihood of the estimated parameters of the GARCH models are given in Table 2. More precisely, in the first part of the table, the estimated parameters of the GARCH models for the period January 1st, 1998, to January 1st, 1999, are given, while in the second part of the Table 2 the estimated parameters of the GARCH models during the period of the economic crisis in 2008 are given.

TABLE 2

THE ESTIMATES OF THE PARAMETERS OF APPROPRIATE GARCH(1,1) MODEL DURING REGULAR MARKET CONDITIONS

Type of GARCH model	GARCH(1,1) with Student's t	GARCH(1,1) with GED	GARCH(1,1) with Skewness GED
	0.066 (0.004)	0.082 (0.001)	0.081 (0.003)
	0.873 (0.000)	0.851 (0.000)	0.860 (0.000)
	0.000 (0.023)	0.000 (0.025)	0.000 (0.027)
			-0.113 (0.045)
η	7.754 (0.000)	1.479 (0.000)	1.515 (0.000)
Log- likelihood	2322.90	2318.08	2320.70
During conditions of the crisis			
Type of GARCH model	GARCH(1,1) with Student's t	GARCH(1,1) with GED	GARCH(1,1) with Skewness GED
	0.114 (0.000)	0.107 (0.000)	0.105 (0.000)
	0.899 (0.000)	0.895 (0.000)	0.895 (0.000)
	0.000 (0.226)	0.000 (0.137)	0.000 (0.133)
			-0.116 (0.240)
η	4.610 (0.000)	1.166 (0.000)	1.180 (0.000)
Log- likelihood	2378.93	2381.70	2385.40

Notes: p-values are given in parentheses

Presented models did not produce any ES breaks, which implies that the models potentially can be reliably used to assess market risk according to the requirements of the Basel III standard. However, this conclusion can only be made based on backtesting. Unlike VaR backtesting, ES backtesting is significantly more complex (Doncic *et al*. 2022). This is the reason why the Basel III standard is not the prescribed manner of backtesting the validity of ES assessments. For that purpose, the in this paper we used two tests Berkowitz's test (LRB) (2001) and Acerbi and Szekely's (2014) first method (Z1). Berkowitz (2001) presented a test based on the Levy Rosenblatt transformation that can be mathematically presented as follows:

(17) $$LR_B = 2\left[\ln L\left(\mu = \mu_{ML}, \sigma^2 = \sigma_{ML}\right) - \ln L\left(\mu = 0, \sigma^2 = 1\right)\right].,$$

where LR_B is the Berkowitz's likelihood ratio. Berkowitz's ES back test is the test that tests a joint hypothesis of zero mean (μ) and unit variance (σ^2), while ($\hat{\mu}_{ML}$) and ($\hat{\sigma}_{ML}$) are (μ) and (σ^2) estimates obtained using maximum likelihood.

The LR_B test is asymptotically distributed as χ2 with two degrees of freedom. Berkowitz's test compares the shape of the forecasted tail of density to the observed tail. Any observations that did not fall within the tail were truncated, noting that the threshold was defined as follows: $TH = max\{ES_1, ES_2, \dots ES_t\}$. Since the Berkowitz test validity is disputed in the case of a relatively small number of exceedances, one of the authors in (Radivojevic *et al*. 2019) proposed the use of bootstrap simulation, where F is the unknown distribution of the estimator θ^k . Thus, Berkowitz's ES backtesting based on bootstrap simulation as presented (Radivojevic *et al*., 2019) was used in the paper. In fact, the estimation of the unknown density F of our ES estimates was used by repeating the simulations by the appropriate models several times[1]. The number of bootstrap repetitions is determined according to the Andrews and Buchinsky procedure (Andrews and Buchinsky, 1997). Determining the bootstrap repetitions number is particularly important in this case because the sample of the breaks utilized in obtaining a single ES estimate is a small fraction of the number of draws. The procedure for calculating the p-value is then continued by analogy, as previously described. The results are given in Table 3.

Given the limitations of Berkowitz's test, Acerbi and Szekely's first method was also used to test the model's validity. Acerbi and Szekely defined the null hypothesis: $H_0 : P_t^{[a]} = F_t^{[a]}$ for $\forall(t)$ against the alternatives $H_1 : \widehat{ES_{\alpha,t}}(X) \geq ES_{\alpha.t}(X) for \forall(t) and > for some(t)$ $\widehat{VaR_{\alpha,t}}(X) \geq VaR_{\alpha.t}(X)$

[1] According to the bootstrap method (Efron and Tibshirani, 1993), we generated multiple new samples from the data sample and calculated the value of the estimator θ^k in each sample. The size of the data-sample, of the exceedances, is known as it is a direct function of the number of trials in the bootstrap simulation and the probability level used in defining the ES. We have chosen a level of error PDB equal to 10% and a confidence level equal to 95%, the initial value of bootstrap repetitions, initial excess kurtosis of the sample of ES repetitions set to zero.

$for \forall(t)$ wherein Ft is the realized distribution of returns, $P_t^{[a]}$ is the conditional distribution tail of the distribution of P_t below the quantile α. We can write this as $P_t^{[a]}(x) = \min\left(1, P_t(x)/a\right)$.

$\widehat{ES}_{\alpha,t}(X)$ and $\widehat{VaR}_{\alpha,t}(X)$ are the sample ES and VaR from the realized returns. Under the null hypothesis, the realized tail is assumed to be the same as the predicted tail of the return distribution. The alternative hypothesis rejects the ES without rejecting VaR. To test the null hypothesis, Acerbi and Szekely defined the following test statistics:

(18) $$Z_1\left(\boldsymbol{X}\right) = \frac{\sum_t^T (X_t I_t / ES_{a,t})}{N_t} + 1$$

where **X** denotes the vector of realized returns ($X_1, X_2, \dots, X_T$), I_t – the indicator function $I_t = 1_{(R_T < VaR_a(R))}$ that indicates the backtesting exceedance of VaR for the realized return X_t in the period *t*, and $N_T = \sum_{t=1}^T I_t$ is the number of the exceedances.

The simulations from the distribution under H0 were used to test for significance in the above method. More precisely, we followed the steps below:

1) simulate X_t^i from P_t for all *t* and *i* = 1,2…, M; where M is a suitably large number of scenarios.

2) for every *i*, compute $Z^i = Z\left(X^i\right)$,i.e., compute the value of Z_1 using the simulations from the first step;

3) estimate the *p*-value as $p = \sum_{i=1}^{M} \frac{Z^i < Z(x)}{M}$, where *Z*(x) is the observed value on Z_1.

4) we conducted the test for a confidence level of 95%.

The results of this test are shown in Table 3.

TABLE 3
BACKTESTING RESULTS DURING REGULAR MARKET CONDITIONS

	QHO	GARCH(1,1) with Student's t	GARCH(1,1) with GED	GARCH(1,1) with Skewness GED
LR_B	0.123	0.210	0.119	0.301
Z_1	0.144	0.172	0.144	0.106
RMSE	0.038	0.052	0.046	0.042
Backtesting results during conditions of the crisis				
	QHO	GARCH(1,1) with Student's t	GARCH(1,1) with GED	GARCH(1,1) with Skewness GED
LR_B	0.154	0.056	0.211	0.177
Z_1	0.122	0.098	0.381	0.428
RMSE	0.075	0.156	0.107	0.082

The p-values were obtained by applying 10.000 simulations. The test was conducted for a confidence level of 95%. According to the results shown in Table 3, it can be concluded that all models successfully passed both tests. Interestingly, no cluster of ES breaks was recorded in any simulations. To answer the question of whether the model contributes to improving the risk assessment, the root mean-squared error ($RMSE = \sqrt{\frac{\sum_{i=1}^{255}\left|R_i^2 - ES_i^2\right|}{255}}$) was used to compare the model performances with the performances of selected models. RMSE results are given also in Table 3. Based on the RMSE results, it can be clearly seen that the model generates smaller deviations, which means smaller capital burdens for banks. Hence, it can be concluded that the model contributes to the improvement of the traditional ES model. These results were obtained under regular market conditions. In the conditions of the crisis, the results are shown in the second part of Table 3. The results, also show that the model generates better risk assessments. This means that the model contributes to the improvement of traditional models and conditions of high volatility. A comparison of models has been also performed in the context of the RMSE of the first four moments of the distribution. The results are given in Table 4 . The results show that the model produces better risk estimates in all four moments of the distribution compared to traditional models. According to Radivojevic et al. (2019), it is clear that in the case of the GARCH model, the assumption of innovations distribution is more important than model specification. In other words, the assumption that is more compatible with real conditions leads to a better-performing model. Since the Student t distribution has a higher degree of freedom parameter than the GED distribution, it was expected to better capture the kurtosis of the return's distribution, especially in crisis conditions, which means that the GARCH(1,1)-Student t model is better equipped to handle extreme events in the data. On the other hand, the GED distribution is better suited for modeling skewness because it has a flexible shape that can be skewed in either direction. This is because the Student t distribution has a symmetric shape, whereas the GED distribution allows for skewness. In situations where the data exhibits skewness, the GARCH(1,1)-GED model may provide better estimates of the dispersion of stock returns. However, theoretical distributions are not fully able to capture empirical phenomena. As the number of extreme cases increases, different variants of Garch models produce larger deviations. They are less able to predict the probability of extreme returns occurring, as well as the magnitude of these deviations. However, it is characteristic of the used variants of the Garch model that it is not possible to make a universal conclusion about which variant is better from the aspect of smaller deviation in moments of the distribution. It is evident that their performance weakens in crisis conditions, but a general conclusion cannot be drawn about whose performance will weaken the most. On the other hand, in the case of the QHO model, the finding showed that it better captures the occurrence of extreme outliers, as well as the probability of their occurrence. This is from reasons because the quantum oscillator provides information about the current trend and momentum of the security.

TABLE 4

THE RESULTS OF COMPARISON OF MODELS IN TERMS OF THE RMSE OF THE FIRST FOUR MOMENTS OF THE DISTRIBUTION

	QHO	S&P Garch(1,1)-student t	S&P Garch(1,1)-GED	S&P Garch(1,1)-Skewed GED
Mean	5.25E-05	1.89E-04	5.47E-05	1.30E-04
Standard Deviation	0.007	0.004	0.011	0.011
Kurtosis	1.104	1.652	1.726	2.080
Skewness	0.801	0.914	0.906	0.924
Conditions of crisis				
	QHO	S&P Garch(1,1)-student t	S&P Garch(1,1)-GED	S&P Garch(1,1)-Skewed GED
Mean	5.37E-05	1.99E-04	5.50E-05	1.40E-04
Standard Deviation	0.011	0.011	0.011	0.011
Kurtosis	1.297	1.865	1.977	2.291
Skewness	0.920	0.921	0.947	0.929

7. Conclusion

First order of time dependent perturbation theory is applied on QHO model of stock market returns for the market with non-Gaussian properties. Semiempirical approach is introduced, since perturbating potential is unknown, to obtain series of time dependent coefficients of QHO wave functions. Fitting procedure over Hermitian polynomial as the independent variables is introduced and enables good fit of empirical data within all second year period of stock returns. In summary, this method enables prediction of tomorrow's outcomes of stock return. Real tomorrow's outcomes show no fallout from predicted ranges within confidence interval of one standard deviation.

In the context of meeting the model validation rules defined by the Basel III standard, the model was tested using Berkowitz's ES backtesting based on bootstrap simulation and Acerbi and Szekely's first method. The model provided satisfactory results. As not only the number of exceedances but also the size of the loss is relevant for the bank, it is important to allow for this criterion when comparing the models. Unfortunately, due to the scope of the work, no such comparison was made with other ES models. For the results to be comparable, for this reason, the data used by Christoffersen were taken.

Model presented in the present paper uses Harmonic oscillator potential, which tends to return the position of the particle towards equilibrium state. In economic terms this implies that stock-markets have the ability of self-correction of stock market returns toward equilibrium. At first glance, it seems that the model can be applied onto markets which are autocorrelated. This would be true for the classical model of the harmonic oscillator, where simple random movement around equilibrium position is described. Considering QHO, this problem is removed, since arbitrary deviation from equilibrium state can be described, even oscilations around new equilibrium state. This can be done by taking into account superposition quantum states, where displacements from equilibrium, which corresponds to pure ground state of QHO can be described with higher quantum states of the particle. Higher QM states are described with higher members of the wave function, i.e. higher orders of Hermite polynomial members in equation 3. For very unstable markets, where stock returns have a long-term tendency of increasing or decreasing, higher order polynomial members will have greater contribution, and larger values of coefficients in Table 1. From a theoretical point of view any stock return distribution can be expanded over infinite numbers of Hermite polynomial members. The only practical question is how fast convergence will occur, and after which order of member series infinite expansion can be truncated. For the crisis period, it was shown that taking into account the first 10 members, stock returns can be well modeled.

Further validation of presented model requires its application on developed and undeveloped markets, so that its general applicability can be verified. It is also important to compare time series of expansion coefficients with the same obtained using different markets in order to check if there is some universal pattern. This will imply that there is some common perturbating potential that can be applied on various stock markets.

The existence of agents whit heterogeneous beliefs and behavioral rules, which may change over time due to social interaction and evolutionary selection, points to the need to respect the views of Dieci and Xe (2018) and Adam *et al*. (2016). The consequence is that their expectations may be different than what would be expected under the assumption of a rational investor. This have to be considered when determining the factors that affect the volatility of returns. For this reason, future research into the application of the quantum oscillator should include how to incorporate these factors into the model.

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